Self-avoiding fractional Brownian motion - The Edwards model

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Abstract

In this work we extend Varadhan's construction of the Edwards polymer model to the case of fractional Brownian motions in $\mathbb{R}^d$, for any dimension $d \geq 2$, with arbitrary Hurst parameters $H \leq 1/d$.

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1 Introduction

In recent years the fractional Brownian motion has become an object of intense study due to its special properties, such as short/long range dependence and self-similarity, leading to proper and natural applications in different fields. In particular, the specific properties of fractional Brownian motion paths have been used e.g. in the modelling of polymers. For the self-intersection properties of sample paths see e.g. [GRV03], [HN05], [HN07], [HNS08], [Ros87], and for the intersection properties with other independent fractional Brownian motion see e.g. [NOL07], [OSS10] and references therein. Comments on the relevance of fractional Brownian motion for polymer modelling, in particular with $H = 1/3$ for polymers in a compact or collapsed phase, can e.g. be found in [BC95].

The fractional Brownian motion on $\mathbb{R}^d$, $d \geq 1$, with Hurst parameter $H \in (0,1)$ is a d-dimensional centered Gaussian process $B^H = \{B_t^H : t \geq 0\}$ with covariance function

$$\mathbb{E}(B_t^{H,i} B_s^{H,j}) = \frac{\delta_{ij}}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad i,j = 1,\ldots,d, \; s,t \geq 0.$$

An informal but suggestive definition of self-intersection local time of a fractional Brownian motion B^H is given in terms of an integral over a Dirac δ-function

$$L = \int_0^T dt \int_0^T ds\, \delta(B^H(t) - B^H(s)),$$

intended to measure the amount of time the process spends intersecting itself in a time interval $[0,T]$. A rigorous definition may be given by approximating the δ-function by the heat kernel

$$p_\varepsilon(x) := \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-\frac{|x|^2}{2\varepsilon}}, \quad x \in \mathbb{R}^d, \varepsilon > 0,$$

which leads to the approximated self-intersection local time

$$L_\varepsilon := \int_0^T dt \int_0^t ds\, p_\varepsilon(B^H(t) - B^H(s)). \tag{1}$$

The main problem is then the removal of the approximation, that is, $\varepsilon \searrow 0$.

In the classic Brownian motion case ($H = 1/2$), L_ε converges in L^2 only for $d = 1$. To ensure the existence of a limiting process for higher dimensions

one must center the approximated self-intersection

$$L_{\varepsilon,c} := L_\varepsilon - \mathbb{E}(L_\varepsilon). \tag{2}$$

For the case of the planar Brownian motion this is sufficient to ensure the L^2-convergence of (2) as ε tends to zero [Var69], but for $d \geq 3$ a further multiplicative renormalization $r(\varepsilon)$ is required to yield a limiting process, now as a limit in law of

$$r(\varepsilon)\left(L_\varepsilon - \mathbb{E}(L_\varepsilon)\right). \tag{3}$$

Through a different approximation, this has been shown in [CY87], [Yor85].

Extending Varadhan's results to the planar fractional Brownian motion, Rosen in [Ros87] shows that, for $1/2 < H < 3/4$, the centered approximated self-intersection local time converges in L^2 as ε tends to zero.

This result, as well as all the above quoted ones for the classic Brownian motion, have been extended by Hu and Nualart in [HN05] to any d-dimensional fractional Brownian motion with $H < 3/4$. More precisely, Hu and Nualart have shown that for $H < 1/d$ the approximated self-intersection local time (1) always converges in L^2. For $1/d \leq H < 3/(2d)$, a L^2-convergence result still holds, but now for the centered approximated self-intersection local time (2). In this case,

$$\mathbb{E}(L_\varepsilon) = \begin{cases} TC_{H,d}\varepsilon^{-d/2+1/(2H)} + o(\varepsilon), & \text{if } 1/d < H < 3/(2d) \\ \frac{T}{2H(2\pi)^{d/2}}\ln(1/\varepsilon) + o(\varepsilon), & \text{if } H = 1/d \end{cases}, \tag{4}$$

where $C_{H,d}$ is a positive constant which depends of H and d. In particular, for $1/d \leq H < \min\{3/(2d), 2/(d+1)\}$, an explicit integral representation for the mean square limiting process L_c as an Itô integral is even obtained in [HNS08]. For $3/(2d) \leq H < 3/4$, a multiplicative renormalization factor $r(\varepsilon)$ is required in [HN05] to prove the convergence in distribution of the random variable (3) to a normal law as ε tends to zero.

To model polymers by Brownian paths Edwards [Edw65] proposed to suppress self-intersections by a factor

$$\exp(-gL),$$

with $g > 0$. For planar Brownian motion Varadhan [Var69] showed that the expectation value $\mathbb{E}(L_\varepsilon)$ has a logarithmic divergence but after its subtraction the centered $L_{\varepsilon,c}$ converges in L^2, with a suitable rate of convergence. From

this, Varadhan could conclude the integrability of $\exp(-gL_c)$, thus giving a proper meaning to the Edwards model. For more details see also [Sim74]. In the three-dimensional case this is clearly much more difficult [Bol93], [Wes80].

In this note we extend Varadhan's construction to arbitrary spatial dimension $d \geq 2$ and Hurst parameters $H \leq 1/d$. For this, the convergence results proved in [HN05] will be essential. Because of this, in the following section we collect from [HN05] the necessary information on fractional Brownian motion and its self-intersection local time, and in Section 3 we state and prove the existence theorem (Theorem 2).

2 Preliminaries

As shown in [HN05], given a d-dimensional fractional Brownian motion B^H with Hurst parameter $H \in (0,1)$, for each $\varepsilon > 0$ the approximated self-intersection local time (1) is a square integrable random variable with

$$\mathbb{E}(L_\varepsilon^2) = \frac{1}{(2\pi)^d} \int_{\mathcal{T}} d\tau \, \frac{1}{((\lambda+\varepsilon)(\rho+\varepsilon)-\mu^2)^{d/2}},$$

where

$$\mathcal{T} := \{(s,t,s',t') : 0 < s < t < T, 0 < s' < t' < T\}$$

and for each $\tau = (s,t,s',t') \in \mathcal{T}$,

$$\lambda(\tau) := (t-s)^{2H}, \quad \rho(\tau) := (t'-s')^{2H}, \tag{5}$$

and

$$\mu(\tau) := \frac{1}{2}\left[|s-t'|^{2H} + |s'^{2H} - t|^{2H} - |t-t'|^{2H} - |s-s'|^{2H}\right]. \tag{6}$$

Furthermore, for each $\varepsilon, \gamma > 0$ is

$$\mathbb{E}(L_\varepsilon L_\gamma) - \mathbb{E}(L_\varepsilon)\mathbb{E}(L_\gamma) = \tag{7}$$

$$\frac{1}{(2\pi)^d} \int_{\mathcal{T}} d\tau \left(\frac{1}{((\lambda+\varepsilon)(\rho+\gamma)-\mu^2)^{d/2}} - \frac{1}{((\lambda+\varepsilon)(\rho+\gamma))^{d/2}} \right) =: E_{\varepsilon\gamma}. \tag{8}$$

Note that the integral in (8) is also well-defined for all $\varepsilon, \gamma \geq 0$ (however it might be infinite). Hence, using this integral representation, we can extend $E_{\varepsilon\gamma}$ to general $\varepsilon, \gamma \geq 0$. This is contrast with (7) which in general is not well-defined for $\varepsilon = 0$ and/or $\gamma = 0$.

From (8) one can easily derive that a necessary and sufficient condition for convergence of $L_{\varepsilon,c} = L_\varepsilon - \mathbb{E}(L_\varepsilon)$ to a limiting process L_c in L^2 as $\varepsilon \searrow 0$ is that $E_{00} < \infty$. As shown in [HN05, Lemma 11], the integral E_{00} is finite if and only if $dH < 3/2$.

3 Theorems and Proofs

Theorem 1 *Assume that* $(d+1)H < 3/2$, $d \geq 2$. *Then there exists a positive constant* K *such that*

$$\mathbb{E}\left((L_{\varepsilon,c} - L_c)^2\right) \leq K\varepsilon^{1/2}$$

for all $\varepsilon > 0$.

Proof. Using (8), a simple calculation and taking the limit $\gamma \searrow 0$ yields

$$\mathbb{E}\left((L_{\varepsilon,c} - L_c)^2\right) = (E_{\varepsilon\varepsilon} - E_{\varepsilon 0}) + (E_{00} - E_{\varepsilon 0})$$

with

$$E_{\varepsilon\varepsilon} - E_{\varepsilon 0} = \frac{d}{2(2\pi)^d} \int_{\mathcal{T}} d\tau\, (\lambda + \varepsilon) \int_0^\varepsilon dx \left(\frac{1}{((\lambda+\varepsilon)(\rho + x))^{d/2+1}} - \frac{1}{((\lambda+\varepsilon)(\rho+x) - \mu^2)^{d/2+1}} \right) \leq 0.$$

Hence

$$\begin{aligned}\mathbb{E}\left((L_{\varepsilon,c} - L_c)^2\right) &\leq E_{00} - E_{\varepsilon 0} \\ &= \frac{d}{2(2\pi)^d} \int_{\mathcal{T}} d\tau\, \rho \int_0^\varepsilon dx \left(\frac{1}{(\delta + x\rho)^{d/2+1}} - \frac{1}{((\lambda + x)\rho)^{d/2+1}} \right), \quad (9)\end{aligned}$$

where $\delta := \lambda\rho - \mu^2$. Thus it is sufficient to establish a suitable upper bound for (9). Technically, this will follow closely the proof of Lemma 11 in [HN05], based on the decomposition of the region $\mathcal{T}$ into three subregions

$$\mathcal{T} \cap \{s < s'\} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3,$$

where

$$\mathcal{T}_1 := \{(t,s,t',s') : 0 < s < s' < t < t' < T\},$$
$$\mathcal{T}_2 := \{(t,s,t',s') : 0 < s < s' < t' < t < T\},$$
$$\mathcal{T}_3 := \{(t,s,t',s') : 0 < s < t < s' < t' < T\}.$$

Each substitution of $\mathcal{T}$ in (9) by a subregion $\mathcal{T}_i$, $i = 1, 2, 3$, yields a different case and for each particular case we will then establish a suitable upper bound.

As in [HN05], we will denote by k a generic positive constant which may be different from one expression to another one. We set $D := d + 1$.

Subregion $\mathcal{T}_1$: We do the change of variables $a := s' - s$, $b := t - s'$, and $c = t' - t$ for $(t,s,t',s') \in \mathcal{T}_1$. Thus, on $\mathcal{T}_1$, for the functions λ, ρ, and μ defined in (5) and (6) we have

$$\lambda(t,s,t',s') =: \lambda_1(a,b,c) = (a+b)^{2H}, \quad \rho(t,s,t',s') =: \rho_1(a,b,c) = (b+c)^{2H}$$
$$\mu(t,s,t',s') =: \mu_1(a,b,c) = \frac{1}{2}\left[(a+b+c)^{2H} + b^{2H} - c^{2H} - a^{2H}\right].$$

On the region $\mathcal{T}_1$ one can bound (9) by the first term only, and to estimate the latter we shall use Lemma 3 below, yielding

$$\rho_1 \int_0^\varepsilon dx \, \frac{1}{(\delta_1 + x\rho_1)^{(D+1)/2}} \le A\varepsilon^{1/2}\rho_1^{1/2}\delta_1^{-D/2}.$$

From [HN05, eq. (59)],

$$\delta_1 \ge k(a+b)^H(b+c)^H a^H c^H \ge k(abc)^{4H/3},$$

we deduce

$$\int_{[0,T]^3} da\, db\, dc\, \delta_1^{-D/2} \le k \int_{[0,T]^3} da\, db\, dc\, (abc)^{-2DH/3} < \infty,$$

because $DH < 3/2$. In conclusion the part of (9) stemming from integration over $\mathcal{T}_1$ is of order $\varepsilon^{1/2}$.

On the subregions $\mathcal{T}_i$, $i = 2, 3$, we have to consider the difference

$$\Xi_i^\varepsilon := \rho_i \int_0^\varepsilon dx \left(\frac{1}{(\delta_i + x\rho_i)^{(D+1)/2}} - \frac{1}{((\lambda_i + x)\rho_i)^{(D+1)/2}} \right), \quad \varepsilon > 0.$$

Subregion $\mathcal{T}_2$: In this case we do the change of variables $a := s' - s$, $b := t' - s'$, and $c = t - t'$ for $(t, s, t', s') \in \mathcal{T}_2$. That is, on $\mathcal{T}_2$ we will have

$$\lambda(t, s, t', s') =: \lambda_2(a, b, c) = b^{2H}, \quad \rho(t, s, t', s') =: \rho_2(a, b, c) = (a + b + c)^{2H}$$
$$\mu(t, s, t', s') =: \mu_2(a, b, c) = \frac{1}{2}\left[(b + c)^{2H} + (a + b)^{2H} - c^{2H} - a^{2H}\right].$$

In this case we decompose the corresponding integral (9) over the regions $\{b \geq \eta a\}$, $\{b \geq \eta c\}$, and $\{b < \eta a, b < \eta c\}$, for some fixed but arbitrary $\eta > 0$. We have by (16), see Appendix,

$$\begin{aligned}\int_{b\geq\eta a} da\, db\, dc\, \Xi_2^\varepsilon &\leq C\varepsilon^{1/2} \int_{b\geq\eta a} da\, db\, dc\, \rho_2^{1/2} \left(\lambda_2\rho_2\right)^{-D/2} \\ &\leq k\varepsilon^{1/2} \int_{b\geq\eta a} \frac{da\, db\, dc}{(a + b + c)^{DH} b^{DH}}.\end{aligned}$$

If $DH < 1$, the integral is finite. If $1 < DH < 3/2$, then by Young inequality

$$\begin{aligned}\int_{b\geq\eta a} da\, db\, dc\, \Xi_2^\varepsilon &\leq k\varepsilon^{1/2} \int_0^T \int_0^T \frac{da\, dc}{(a + c)^{DH}} \int_{\eta a}^T db\, b^{-DH} \\ &\leq k\varepsilon^{1/2} \int_0^T da\, a^{-4DH/3+1} \int_0^T dc\, c^{-2DH/3} < \infty.\end{aligned}$$

In the case $DH = 1$ we have

$$\int_{b\geq\eta a} da\, db\, dc\, \Xi_2^\varepsilon \leq k\varepsilon^{1/2} \int_0^T dc\, c^{-2/3} \int_0^T da\, a^{-1/3} \ln(T/(\eta a)) < \infty.$$

The case $b \geq \eta c$ can be treated analogously.

To handle the case $b < \eta a$ and $b < \eta c$ we first observe that

$$\begin{aligned}\mu_2 &= \frac{1}{2}\left(a^{2H}\left(\left(1 + \frac{b}{a}\right)^{2H} - 1\right) + c^{2H}\left(\left(1 + \frac{b}{c}\right)^{2H} - 1\right)\right) \\ &\leq k\left(a^{2H-1} + c^{2H-1}\right) b\end{aligned}$$

for sufficiently small $\eta > 0$. Hence, together with (15), see Appendix, we

obtain

$$
\begin{aligned}
\int_{b<\eta a, b<\eta c} da\, db\, dc\, \Xi_2^\varepsilon \leq C\varepsilon^{1/2} \int_{b<\eta a, b<\eta c} da\, db\, dc\, \rho_2^{1/2} \mu_2^2 \left(\lambda_2 \rho_2\right)^{-(D+2)/2} \\
\leq k\varepsilon^{1/2} \int_{b<\eta a, b<\eta c} da\, db\, dc\, \left(a^{4H-2} + c^{4H-2}\right) (a+b+c)^{-2H-DH} b^{2-2H-DH} \\
\leq k\varepsilon^{1/2} \int_{b<\eta a, b<\eta c} da\, db\, dc\, b^{-DH} (a+b+c)^{-2H-DH} \\
\times \left(a^{(2-D/3)H} b^{DH/3} + c^{(2-D/3)H} b^{DH/3}\right) \\
\leq k\varepsilon^{1/2} \int_{[0,T]^3} da\, db\, dc\, b^{-DH} (a+b+c)^{-2H-DH} a^{(2-D/3)H} b^{DH/3} \\
\leq k\varepsilon^{1/2} \int_{[0,T]^3} da\, db\, dc\, b^{-2DH/3} c^{-2DH/3} a^{-2DH/3} < \infty,
\end{aligned}
$$

because $DH < 3/2$.

Subregion $\mathcal{T}_3$: We do the change of variables $a := t - s$, $b := s' - t$, and $c = t' - s'$ for $(t, s, t', s') \in \mathcal{T}_3$. Thus, on $\mathcal{T}_3$, we have

$$
\begin{aligned}
&\lambda(t, s, t', s') =: \lambda_3(a, b, c) = a^{2H}, \quad \rho(t, s, t', s') =: \rho_3(a, b, c) = c^{2H} \\
&\mu(t, s, t', s') =: \mu_3(a, b, c) = \frac{1}{2} \left[(a+b+c)^{2H} + b^{2H} - (b+c)^{2H} - (a+b)^{2H}\right].
\end{aligned}
$$

In this case we decompose the corresponding integral (9) over the regions $\{a \geq \eta_1 b, c \geq \eta_2 b\}$, $\{a < \eta_1 b, c < \eta_2 b\}$, $\{a \geq \eta_1 b, c < \eta_2 b\}$, and $\{a < \eta_1 b, c \geq \eta_2 b\}$ for some fixed but arbitrary $\eta_1, \eta_2 > 0$. By symmetry it suffices to consider the first three regions. Using (16), see Appendix, we obtain

$$
\begin{aligned}
\int_{a\geq\eta_1 b, c\geq\eta_2 b} da\, db\, dc\, \Xi_3^\varepsilon \leq C\varepsilon^{1/2} \int_{a\geq\eta_1 b, c\geq\eta_2 b} da\, db\, dc\, \rho_3^{1/2} \left(\lambda_3 \rho_3\right)^{-D/2} \\
\leq k\varepsilon^{1/2} \int_0^T db \int_{\eta_1 b}^T \frac{da}{a^{DH}} \int_{\eta_2 b}^T \frac{dc}{c^{DH}} \leq k\varepsilon^{1/2} \int_0^T \frac{db}{b^{2DH-2}} < \infty.
\end{aligned}
$$

For the region $\{a < \eta_1 b, c < \eta_2 b\}$, we observe that since $H < 3/(2D) \leq 1/2$, we can conclude from (15), see Appendix, together with [HN05, eq. (55)], i.e., $\mu_3 \leq k b^{2H-2} ac$, that

$$
\begin{aligned}
\Xi_3^\varepsilon &\leq C\varepsilon^{1/2} \rho_3^{1/2} \mu_3^2 \left(\lambda_3 \rho_3\right)^{-(D+2)/2} \\
&\leq k\varepsilon^{1/2} b^{4H-4} a^{2-2H-DH} c^{2-2H-DH} \leq k\varepsilon^{1/2} a^{-2DH/3} c^{-2DH/3} b^{-2DH/3},
\end{aligned}
$$

which is integrable. Finally, we consider the case $\{a \geq \eta_1 b, c < \eta_2 b\}$. For $\eta_2 > 0$ small enough we have

$$\begin{aligned}\mu_3 &= \frac{1}{2}\left((a+b)^{2H}\left(\left(1+\frac{c}{a+b}\right)^{2H}-1\right)-b^{2H}\left(\left(1+\frac{c}{b}\right)^{2H}-1\right)\right)\\ &\leq k\left((a+b)^{2H-1}+b^{2H-1}\right)c = kb^{2H-1}\left(\left(1+\frac{a}{b}\right)^{2H-1}+1\right)c \leq kb^{2H-1}c,\end{aligned}$$

where in the last estimate we used $2H-1<0$ (due to $H<1/2$). Then using (15) we obtain

$$\begin{aligned}\int_{a\geq\eta_1 b, c<\eta_2 b} da\,db\,dc\,\Xi_3^\varepsilon &\leq C\varepsilon^{1/2}\int_{a\geq\eta_1 b, c<\eta_2 b} da\,db\,dc\,\rho_3^{1/2}\mu_3^2\,(\lambda_3\rho_3)^{-(D+2)/2}\\ &\leq k\varepsilon^{1/2}\int_{a\geq\eta_1 b} da\,db\,b^{4H-2}a^{-2H-DH}\int_0^{\eta_2 b} dc\,c^{2-2H-DH}\\ &\leq k\varepsilon^{1/2}\int_0^T da\,a^{-2H-DH}\int_0^{a/\eta_2} db\,b^{-DH+2H+1} \leq k\varepsilon^{1/2}\int_0^T da\,a^{-2DH+2},\end{aligned}$$

which is finite because $DH < 3/2$. ■

Theorem 2 *(i) Assume that $dH = 1$, $d \geq 2$. Then there exists a positive constant M such that for all $0 \leq g \leq M$*

$$\exp(-gL_c) \tag{10}$$

is an integrable function.
(ii) Assume that $dH < 1$, $d \geq 2$. Then there exists

$$L := \lim_{\varepsilon\searrow 0} L_\varepsilon \ \textit{in}\ L^2$$

and for all non-negative constants g

$$\exp(-gL)$$

is an integrable function.

Proof. (i) The case $d=2$ and $H=1/2$ was treated in [Var69]. In all other cases we are in the situation of Theorem 1. In these cases we have a

logarithmic divergence of $\mathbb{E}(L_\varepsilon)$ as $\varepsilon \searrow 0$, see (4). Combining this moderate divergence with the rate of convergence provided in Theorem 1, the proof for integrability of (10) for small enough non-negative g follows very close along the lines of [Var69, proof of Step 3]. More precisely, by (4), for $0 < \varepsilon \leq 1$ there exists a positive constant k such that

$$L_{\varepsilon,c} \geq -\mathbb{E}(L_\varepsilon) \geq -k - \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)|.$$

For any constant $N \geq k + \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)|$ one has

$$\begin{aligned} \mathbb{P}(L_c \leq -N) &= \mathbb{P}(L_c - L_{\varepsilon,c} \leq -N - L_{\varepsilon,c}) \\ &\leq \mathbb{P}\left(|L_{\varepsilon,c} - L_c| \geq N - k - \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)| \right). \end{aligned}$$

An application of Chebyshev's inequality then yields

$$\mathbb{P}(L_c \leq -N) \leq \frac{\mathbb{E}(|L_{\varepsilon,c} - L_c|^2)}{\left(N - k - \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)|\right)^2} \leq K \frac{\varepsilon^{1/2}}{\left(N - k - \frac{T}{2H(2\pi)^{d/2}} |\ln(\varepsilon)|\right)^2}.$$

In particular, for

$$\varepsilon = \exp\left(-H(2\pi)^{d/2}(N-k)/T\right)$$

one obtains

$$\mathbb{P}(L_c \leq -N) \leq \frac{4K}{(N-k)^2} \exp\left(-H(2\pi)^{d/2}(N-k)/(2T)\right).$$

Hence, there exists a positive constant M such that (10) is integrable for all $0 \leq g \leq M$.

(ii) In the cases $dH < 1$ we know from [HN05, Theorem 1 (i)] that the following limit exists

$$0 \leq L := \lim_{\varepsilon \searrow 0} L_\varepsilon \text{ in } L^2.$$

Thus, $\exp(-gL)$ is integrable for all non-negative g. ■

Appendix

The following lemma is an immediate consequence of the Cauchy–Schwartz inequality.

Lemma 3 *Let $0 < \alpha, \beta < \infty$ and $1/2 < m < \infty$, then there exists a positive constant A such that*

$$\int_0^\varepsilon dx\, (\alpha + \beta x)^{-m} \le A\varepsilon^{1/2}\alpha^{-m+1/2}\beta^{-1/2}.$$

For $i = 2, 3$ we set

$$\xi_i(x) := \frac{1}{(\delta_i + x\rho_i)^{(D+1)/2}} - \frac{1}{((\lambda_i + x)\rho_i)^{(D+1)/2}}, \quad x \ge 0.$$

The following lemma is a generalization of estimates (56) and (57) obtained in [HN05, Lemma 10].

Lemma 4 *For $i = 2, 3$ there exists a positive constant B such that*

$$\xi_i(x) \le B\mu_i^2((\lambda_i + x)\rho_i)^{-(D+1)/2-1}, \tag{11}$$

$$\xi_i(x) \le B\left((\lambda_i + x)\,\rho_i\right)^{-(D+1)/2}, \tag{12}$$

for all $x \ge 0$.

Proof. Estimate (11) implies estimate (12). Indeed, according to [Hu01, Lemma 3 (2)], for some suitable constant $0 < k < 1$,

$$\lambda_i\rho_i - \mu_i^2 = \delta_i \ge k\lambda_i\rho_i.$$

Since λ_i, ρ_i are positive, this implies that

$$\mu_i^2 \le (1-k)\lambda_i\rho_i \le (1-k)(\lambda_i + x)\rho_i, \tag{13}$$

for all $x \ge 0$. Thus, assuming (11), (12) follows from (13).

Therefore, the proof amounts to prove (11). Given

$$\xi_i(x) = \left(\left(1 - \frac{\mu_i^2}{(\lambda_i + x)\rho_i}\right)^{-(D+1)/2} - 1\right)((\lambda_i + x)\rho_i)^{-(D+1)/2} \tag{14}$$

observe that due to (13)

$$0 \le \frac{\mu_i^2}{(\lambda_i + x)\rho_i} \le 1 - k < 1.$$

Hence let us consider the function

$$[0, 1-k] \ni y \mapsto f(y) := (1-y)^{-(D+1)/2} - 1 \in [0, \infty).$$

Since $f(0) = 0$ and f' is continuous on $(0, 1-k)$ with a continuous continuation to $[0, 1-k]$, there exists a positive constant B such that

$$f(y) \leq \max_{z \in [0,1-k]} |f'(z)| y \leq By \quad \text{for all} \quad y \in [0, 1-k].$$

Applying this inequality to (14) yields the required estimate (11). ■

Lemma 5 *For $i = 2, 3$ there exists a positive constant C such that*

$$\Xi_i^\varepsilon \leq C\varepsilon^{1/2} \rho_i^{1/2} \mu_i^2 (\lambda_i \rho_i)^{-(D+2)/2}, \tag{15}$$

$$\Xi_i^\varepsilon \leq C\varepsilon^{1/2} \rho_i^{1/2} (\lambda_i \rho_i)^{-D/2}, \tag{16}$$

for all $\varepsilon > 0$.

Proof. Recall that

$$\Xi_i^\varepsilon = \rho_i \int_0^\varepsilon dx\, \xi_i(x), \quad i = 2, 3.$$

Hence (15) and (16) follow from (11) and (12), respectively, together with Lemma 3. ■

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